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Boundary value representations for bounded hyperfunctions and some variants

By

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Abstract

There are two notions of boundedness for hyperfunctions: the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions and the sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions at infinity. The former was introduced by Chung-Kim-Lee [2] using a duality method, and the latter was introduced by [5] in a cohomological manner, where we also gave an identification between $\mathcal{B}_{L^\infty}$ in one dimensional case and the space of the global sections of $\mathscr{B}_{L^\infty}$. This identification can be regarded as boundary value representations of bounded hyperfunctions in one dimensional case.

In this report, we study bounded hyperfunctions in the general case, and announce our recent result on their boundary value representations by bounded holomorphic functions on wedges with respect to the octant decompositions. We also mention some variants including reflexive-valued cases.

§ 1. Introduction

The notion of hyperfunction was introduced by M. Sato [6], [7], [8], and plays important roles in the study of analytic ordinary and partial differential equations.

Hyperfunctions have many good and convenient properties. They form a flabby sheaf $\mathscr{B}$ on the real euclidean space $\mathbb{R}^n$ (or on a real-analytic manifold), and admit boundary value representations by holomorphic defining functions. Through these boundary value representations, hyperfunctions also admit comparatively direct action of linear differential operators with real-analytic coefficients.

On the other hand, it is also well known that there is no inequality nor boundedness for hyperfunctions. Similarly, no good topology is known on the space $\mathscr{B}(\Omega)$ of hyperfunctions on an open set $\Omega \subset \mathbb{R}^n$.

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Such inconveniences were sometimes overcome by introducing new classes of hyperfunctions. An instance is the notion of Fourier hyperfunctions. In fact, there is no notion of Fourier transformation for $\mathscr{B}(\mathbb{R}^n)$, and the sheaf $\mathscr{Q}$ of Fourier hyperfunctions on a compactification $\mathbb{D}^n := \mathbb{R}^n \sqcup S^{n-1}$ was constructed by [6] in case $n = 1$, and by T. Kawai [3] in the general case, in order to introduce Fourier analysis for hyperfunctions. Their definitions are cohomological, but the space $\mathscr{Q}(\mathbb{D}^n)$ of global sections can be identified with the dual space of a suitable space $\mathcal{P}_*$ of exponentially decaying test functions.

In [2], S. Y. Chung, D. Kim and E. G. Lee introduced the notion of boundedness for hyperfunctions; they constructed the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions in several variables. Their definition was given by duality; actually, they defined the spaces $\mathcal{B}_{L^p}$ of hyperfunctions of L^p growth for $1 < p \leq \infty$, as the dual space of suitable test function spaces $\mathcal{A}_{L^q}$ with $1/p + 1/q = 1$, $1 \leq q < \infty$. Then, the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions is the variant with respect to $p = \infty$. They gave the standard inclusion $\mathcal{B}_{L^\infty} \hookrightarrow \mathscr{Q}(\mathbb{D}^n)$, by comparing $\mathcal{A}_{L^1}$ with the space $\mathcal{P}_*$. Moreover, they studied $\mathcal{B}_{L^\infty}$ by Matsuzawa's heat kernel method, and show several properties, including the structure theorem, its relation with periodic hyperfunctions, etc.

On the other hand, in [5], we introduced the sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions at infinity in one variable on a compactification $\mathbb{D}^1 = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$, for the purpose of the study of Massera type theorems in hyperfunctions. (Refer to [5] and also to J. L. Massera [4] for Massera type theorems). Our construction is described in terms of boundary value representations by sections of the sheaf $\mathscr{O}_{L^\infty}$ of bounded holomorphic functions on the space $\mathbb{D}^1 + i\mathbb{R}$. We also proved that the space $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ of global sections of our sheaf can be identified with their space $\mathcal{B}_{L^\infty}$ in one-dimensional case, that is, $\mathscr{B}_{L^\infty}(\mathbb{D}^1)$ is isomorphic to the dual space of $\mathcal{A}_{L^1}$.

Since we had not constructed multi-dimensional variants of the sheaf of bounded hyperfunctions at infinity, there is no counterpart of such identification in several variables. But the identification in univariate case can be interpreted as the boundary value representations by $\mathscr{O}_{L^\infty}$ for the duals (continuous linear functionals) of $\mathcal{A}_{L^1}$, which may be multi-dimensionalizable. Also for the purpose of the study of vector valued variants of $\mathscr{B}_{L^\infty}$, it seems important to think about dualities.

For these purposes, we report boundary value representations for $\mathcal{B}_{L^\infty}$ in several variables by defining functions in $\mathscr{O}_{L^\infty}$ on wedges with respect to octant decompositions. Moreover, we introduce the variants of $\mathscr{B}_{L^\infty}$ taking values in a reflexive locally convex space, and give boundary value representations for them.

§ 2. Boundedness for hyperfunctions

Let us recall the definition of the space $\mathcal{B}_{L^\infty}$ of bounded hyperfunctions due to Chung-Kim-Lee [2]. They introduced the spaces of test functions for $1 \le q < +\infty$ by

$$\mathcal{A}_{L^q,h} := \{\varphi \in C^\infty(\mathbb{R}^n); \|\varphi\|_{L^q,h} := \sup_{\alpha\in\mathbb{N}^n} \frac{\|\partial^\alpha\varphi\|_{L^q(\mathbb{R}^n)}}{h^{|\alpha|}\alpha!} < +\infty\}, \tag{2.1a}$$

$$\mathcal{A}_{L^q} := \varinjlim_{h>0} \mathcal{A}_{L^q,h}. \tag{2.1b}$$

Here we used the standard notations of multiindices and derivations: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\partial_i = \partial/\partial x_i$ $(i = 1, \dots, n)$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Note that for each $h > 0$, $\mathcal{A}_{L^q,h}$ becomes a Banach space with the norm $\|\cdot\|_{L^q,h}$ given in (2.1a), and we endow $\mathcal{A}_{L^q}$ with the inductive limit locally convex topology by (2.1b).

Definition 2.1 ($\mathcal{B}_{L^\infty}$). Let $1 < p \le +\infty$ and take q $(1 \le q < +\infty)$ with $1/p+1/q = 1$. The space $\mathcal{B}_{L^p}$ of hyperfunctions with L^p growth is defined as the dual space of $\mathcal{A}_{L^q}$. In particular, $\mathcal{B}_{L^\infty}$ is called the space of bounded hyperfunctions.

Let us also recall the sheaf $\mathscr{B}_{L^\infty}$ of bounded hyperfunctions at infinity defined on a compactification $\mathbb{D}^1 := [-\infty, +\infty] = \mathbb{R} \sqcup \{\pm\infty\}$ of $\mathbb{R}$, introduced in [5]. We consider the topological spaces

$$\begin{array}{ccc} \mathbb{C} = \mathbb{R} + i\mathbb{R} & \subset & \mathbb{D}^1 + i\mathbb{R} \\ \cup & & \cup \\ \mathbb{R} =]-\infty, +\infty[& \subset & \mathbb{D}^1 = [-\infty, +\infty] \end{array}$$

and take coordinates $t \in \mathbb{R}$ and $w \in \mathbb{C}$, $(\mathrm{Re}\, w = t)$. The sheaf of holomorphic functions on $\mathbb{C}$ is denoted by $\mathscr{O}$.

Definition 2.2 ($\mathscr{O}_{L^\infty}$). The sheaf $\mathscr{O}_{L^\infty}$ of bounded holomorphic functions on $\mathbb{D}^1 + i\mathbb{R}$ is defined as the sheaf associated with the presheaf

$$\mathbb{D}^1 + i\mathbb{R} \supset U \mapsto \mathscr{O}(U \cap \mathbb{C}) \cap L^\infty(U \cap \mathbb{C}).$$

We have the following facts:

- $\mathscr{O}_{L^\infty}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}) \mid \forall K \Subset U, \|f\|_K := \sup_{w\in K\cap\mathbb{C}}|f(w)| < +\infty\}$.
- $\mathscr{O}_{L^\infty}(U)$ is a Fréchet space.
- $\mathscr{O}_{L^\infty}|_{\mathbb{C}} = \mathscr{O}$, that is, $\mathscr{O}_{L^\infty}(U) = \mathscr{O}(U)$ if $U \subset \mathbb{C}$.

Definition 2.3 ($\mathscr{B}_{L^\infty}$). The sheaf $\mathscr{B}_{L^\infty}$ of *bounded hyperfunctions at infinity* on $\mathbb{D}^1$ is defined as the sheaf associated with the presheaf given by the correspondence

$$\mathbb{D}^1 \overset{\text{open}}{\supset} \Omega \mapsto \varinjlim_{U} \frac{\mathscr{O}_{L^\infty}(U \setminus \Omega)}{\mathscr{O}_{L^\infty}(U)}.$$

Here U runs through complex neighborhoods of Ω, and an open set $U \subset \mathbb{D}^1 + i\mathbb{R}$ is called a complex neighborhood of a locally closed set $\Omega \in \mathbb{D}^1$ if Ω is included in U as a closed subset.

Under the notations $B_d :=]-d, d[$, $\dot{B}_d :=]-d, d[\setminus \{0\}$ for $d > 0$, the space of the global sections of $\mathscr{B}_{L^\infty}$ can be expressed as:

$$\mathscr{B}_{L^\infty}(\mathbb{D}^1) \simeq \varinjlim_{d>0} \frac{\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)}{\mathscr{O}_{L^\infty}(\mathbb{D}^1 + iB_d)}. \tag{2.2}$$

Let E be a sequentially complete Hausdorff locally convex space. Then, the notion of E-valued holomorphic function makes sense. (Refer for example to [1] for holomorphic functions taking values in a local convex space.) Starting from the sheaf ${}^E\mathscr{O}$ of E-valued holomorphic functions on $\mathbb{C}$, we can define sheaves ${}^E\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ and ${}^E\mathscr{B}_{L^\infty}$ on $\mathbb{D}^1$ in a parallel manner.

Definition 2.4 (${}^E\mathscr{B}_{L^\infty}$)**.** We define the sheaf ${}^E\mathscr{O}_{L^\infty}$ on $\mathbb{D}^1 + i\mathbb{R}$ as the sheaf associated with the presheaf

$$U \mapsto \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}) \mid f \text{ is bounded.}\},$$

and also ${}^E\mathscr{B}_{L^\infty}$ on $\mathbb{D}^1$ as the sheaf associated with the presheaf

$$\Omega \mapsto \varinjlim_{U} \frac{{}^E\mathscr{O}_{L^\infty}(U \setminus \Omega)}{{}^E\mathscr{O}_{L^\infty}(U)}.$$

Similar to the scalar valued case (2.2), the space of the global sections can also be expressed as:

$${}^E\mathscr{B}_{L^\infty}(\mathbb{D}^1) \simeq \varinjlim_{d>0} \frac{{}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + i\dot{B}_d)}{{}^E\mathscr{O}_{L^\infty}(\mathbb{D}^1 + iB_d)}. \tag{2.3}$$

§ 3. Boundary value representations for $\mathcal{B}_{L^\infty}$

We extend the notion of bounded holomorphic functions of one variable to the case of several variables and also to that of L^p growth $(1 \le p \le +\infty)$.

Let $\mathbb{D}^n := \mathbb{R}^n \sqcup S^{n-1}_\infty$ be a compactification of $\mathbb{R}^n$ with the $(n-1)$ dimensional sphere at infinity, and consider the topological spaces

$$\begin{array}{ccc} \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n & \subset & \mathbb{D}^n + i\mathbb{R}^n \\ \cup & & \cup \\ \mathbb{R}^n & \subset & \mathbb{D}^n \end{array}$$

with coordinates $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, $(\mathrm{Re}\, w = t)$. We define the sheaf $\mathscr{O}_{L^p}$ on $\mathbb{D}^n + i\mathbb{R}^n$, and give another description of the test function spaces in (2.1). (See Lemma 3.4 below.)

Definition 3.1 ($\mathscr{O}_{L^p}$). For $1 \le p \le +\infty$, we define the sheaf $\mathscr{O}_{L^p}$ on $\mathbb{D}^n + i\mathbb{R}^n$ as the sheaf associated with the presheaf

$$\mathbb{D}^n + i\mathbb{R}^n \supset U \mapsto \mathscr{O}(U \cap \mathbb{C}^n) \cap L^p(U \cap \mathbb{C}^n).$$

Note that Definition 2.2 is a special case with $n = 1$ and $p = \infty$ of Definition 3.1. We can show the following facts.

- $\mathscr{O}_{L^p}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}^n) \mid \forall K \Subset U, \|f\|_{L^p,K} < +\infty\}$. Here and in what follows, we use the abbreviation $\|f\|_{L^p,K}$ for $\|f\|_{L^p(K\cap\mathbb{C}^n)}$.
- $\mathscr{O}_{L^p}|_{\mathbb{C}} = \mathscr{O}$, that is, $\mathscr{O}_{L^p}(U) = \mathscr{O}(U)$ if $U \subset \mathbb{C}^n$.
- $\mathscr{O}_{L^p}(U) \subset \mathscr{O}_{L^q}(U)$ if $1 \le p \le q \le +\infty$.

We endow $\mathscr{O}_{L^p}(U)$ with a locally convex topology given by a family of semi-norms $\{\|\cdot\|_{L^p,K}\}_{K\Subset U}$.

Moreover, we introduce another type of semi-norms:

$$\|f\|_{L^\infty L^p,K} := \sup_{s\in\mathbb{R}^n} \|(\chi_{K\cap\mathbb{C}^n} f)(\cdot + is)\|_{L^p(\mathbb{R}^n)}.$$

Here $\chi_{K\cap\mathbb{C}^n}$ denotes the characteristic function of $K \cap \mathbb{C}^n$ and consider $\chi_{K\cap\mathbb{C}^n} f$ as a function on $\mathbb{C}^n$ with value 0 outside $K \cap \mathbb{C}^n$.

This family $\{\|\cdot\|_{L^\infty L^p,K}\}_{K\Subset U}$ defines the same subspace $\mathscr{O}_{L^p}(U)$ in $\mathscr{O}(U \cap \mathbb{C}^n)$ with the same locally convex topology, as $\{\|\cdot\|_{L^p,K}\}_{K\Subset U}$ does.

Lemma 3.2. *$\mathscr{O}_{L^p}(U) = \{f \in \mathscr{O}(U \cap \mathbb{C}^n) \mid \forall K \Subset U, \|f\|_{L^\infty L^p,K} < +\infty\}$ as locally convex spaces.*

Corollary 3.3. *Let $S \subset \mathbb{R}^n$ be an open set. For a tube domain $\mathbb{D}^n + iS$, we have*

$$\mathscr{O}_{L^p}(\mathbb{D}^n + iS) = \{f \in \mathscr{O}(\mathbb{R}^n + iS) \mid \forall S_0 \Subset S, \|f\|_{L^\infty(\mathbb{R}^n+iS_0)} < +\infty, \sup_{s\in S_0} \|f(\cdot + is)\|_{L^p(\mathbb{R}^n)} < +\infty\},$$

as locally convex spaces.

Now we give another description of $\mathcal{A}_{L^q}$ in terms of $\mathscr{O}_{L^q}$, under the notation $(B_d)^n = \prod_{j=1}^n B_d = \{s \in \mathbb{R}^n \mid \max_j |s_j| < d\}$,

Lemma 3.4. *$\mathcal{A}_{L^q} \simeq \varinjlim_{d>0} \mathscr{O}_{L^q}(\mathbb{D}^n + i(B_d)^n)$ as locally convex spaces.*

Consider a pair of indices $1 \le p, q \le +\infty$ with $1/p + 1/q = 1$ and a tube domain $\mathbb{D}^n + iS$ with a connected open set $S \subset \mathbb{R}^n$.

Lemma 3.5. *For $F \in \mathscr{O}_{L^p}(\mathbb{D}^n + iS)$, $f \in \mathscr{O}_{L^q}(\mathbb{D}^n + iS)$, and $s \in S$, the integral $\int_{\mathbb{R}^n} F(t + is) f(t + is)dt$ is well-defined and independent of s.*

Proof. Using the Hölder inequality and the Lebesgue convergence theorem, we have $\|(Ff)(\cdot+is)\|_{L^1(\mathbb{R}^n)} \le \|F(\cdot+is)\|_{L^p(\mathbb{R}^n)}\|f(\cdot+is)\|_{L^q(\mathbb{R}^n)} < +\infty$, and

$$\int_{\mathbb{R}^n} F(t+is)f(t+is)dt = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^n+is} F(w)f(w)e^{-\varepsilon w^2}dw.$$

Since Ff is holomorphic and $Ff \in L^\infty(\mathbb{R}^n+iS_0)$ for any $S_0 \Subset S$, we can deform the contour in $\int_{\mathbb{R}^n+is} F(w)f(w)e^{-\varepsilon w^2}dw$. □

The boundary value representations for $\mathcal{B}_{L^\infty}$ for n-dimensional case is given just in a parallel way as those in [5] for 1-dimensional case.

Fix a constant $d>0$, and we define open sets U, $\dot{U}$, $\dot{U}_j$ $(j=1,\dots,n)$ by

$$\tag{3.1}\begin{cases} U := \mathbb{D}^n + i(B_d)^n, \\ \dot{U} := \mathbb{D}^n + i(\dot{B}_d)^n, \\ \dot{U}_j := \mathbb{D}^n + i(\dot{B}_d\times\cdots\times \underset{j\text{-th}}{B_d}\times\cdots\times\dot{B}_d) \\ \qquad = \{w\in\mathbb{D}^n + i(B_d)^n \mid \operatorname{Im} w_k \neq 0 \text{ if } k\neq j\}. \end{cases}$$

For r with $0<r<d$, we define contours $\gamma(r)$, $\gamma(r,n)$ by

$$\tag{3.2}\gamma(r) := -\partial(\mathbb{R}+iB_r), \quad \gamma(r,n) := \overbrace{\gamma(r)\times\cdots\times\gamma(r)}^{n}.$$

Consider $F\in\mathscr{O}_{L^\infty}(\dot{U})$ and $\varphi\in\mathscr{O}_{L^1}(U)$. We take r with $0<r<d$ and define

$$\langle F,\varphi\rangle := \int_{\gamma(r,n)} F(w)\varphi(w)dw.$$

We can easily see the well-definedness as follows.

Lemma 3.6. *The right hand side is integrable and independent of r.*

Proof. It can be written as

$$\sum_{\epsilon\in\{\pm1\}^n}\operatorname{sgn}(\epsilon)\int_{\mathbb{R}^n} F(t+ir\epsilon)\varphi(t+ir\epsilon)dt.$$

By Lemma 3.5, each integral is well-defined and independent of r. □

We define the boundary value map $b\colon \mathscr{O}_{L^\infty}(\dot{U}) \to \mathcal{B}_{L^\infty} = (\mathcal{A}_{L^1})'$ by

$$\tag{3.3} b(F)(\varphi) = \langle F,\varphi\rangle, \quad \text{for } F\in\mathscr{O}_{L^\infty}(\dot{U}),\ \varphi\in\mathscr{O}_{L^1}(U),$$

and also the defining function map $g:\mathcal{B}_{L^\infty} = (\mathcal{A}_{L^1})' \to \mathscr{O}_{L^\infty}(\dot{U})$ by

$$\tag{3.4} g(\psi)(w) := \psi(K(w-\cdot)), \quad \text{for } \psi\in(\mathcal{A}_{L^1})',\ w\in\dot{U},$$

where

$$K(w) := \frac{1}{(-2\pi i)^n} \cdot \frac{e^{-w^2}}{w_1 \cdots w_n}. \tag{3.5}$$

Then we can show,

- $b(\sum_{j=1}^n \mathscr{O}_{L^\infty}(\dot{U}_j)) = 0$, and therefore b induces a linear map from the quotient space $\mathscr{O}_{L^\infty}(\dot{U})/\sum_{j=1}^n \mathscr{O}_{L^\infty}(\dot{U}_j)$ to $\mathscr{B}_{L^\infty}$.
- $K(w, \cdot)$ belongs to $\mathcal{A}_{L^1}$ for a fixed $w \in \dot{U}$, and therefore $g(\psi)(w)$ is well defined.
- $g(\psi)(w)$ is holomorphic in w, and defines a section $g(\psi) \in \mathscr{O}_{L^\infty}(\dot{U})$.

For $F \in \mathscr{O}_{L^\infty}(\dot{U})$, we denote by $[F]$ the class in $\mathscr{O}_{L^\infty}(\dot{U})/\sum_{j=1}^n \mathscr{O}_{L^\infty}(\dot{U}_j)$ represented by F.

Theorem 3.7. *We have $[g(b(F))] = [F]$ for any $F \in \mathscr{O}_{L^\infty}(\dot{U})$ and $b(g(\psi)) = \psi$ for any $\psi \in \mathcal{B}_{L^\infty}$. Therefore, b induces an isomorphism between vector spaces:*

$$\frac{\mathscr{O}_{L^\infty}(\dot{U})}{\sum_{j=1}^n \mathscr{O}_{L^\infty}(\dot{U}_j)} \xrightarrow{\sim} \mathcal{B}_{L^\infty}.$$

Remark. We also have $\mathscr{O}_{L^p}(\dot{U})/\sum_{j=1}^n \mathscr{O}_{L^p}(\dot{U}_j) \xrightarrow{\sim} \mathcal{B}_{L^p}$ for $1 < p \leq +\infty$. The case $n = 1$ was studied by H. Shima in his master thesis presented to Chiba University, 2010, (in Japanese).

Though $\mathcal{A}_{L^q}$ is defined only for $1 \leq q < +\infty$ in (2.1), we can also define $\mathcal{A}_{L^\infty}$ and construct the boundary value map $b\colon \mathscr{O}_{L^1}(\dot{U}) \to (\mathcal{A}_{L^\infty})'$. But, in this case b does not induce an isomorphism.

§ 4. Vector valued cases

Let E be a sequentially complete Hausdorff locally convex space. The system of continuous semi-norms of E is denoted by $\mathcal{N}(E)$. We define vector valued variants of $\mathscr{O}_{L^p}$, as follows.

Definition 4.1 (${}^E\mathscr{O}_{L^p}$). For $1 \leq p \leq +\infty$, the sheaf ${}^E\mathscr{O}_{L^p}$ on $\mathbb{D}^n + i\mathbb{R}^n$ is defined as the sheaf associated with the presheaf

$$\mathbb{D}^n + i\mathbb{R}^n \supset U \mapsto \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}^n) \mid \forall \rho \in \mathcal{N}(E), \rho \circ f \in L^p(U \cap \mathbb{C}^n)\}.$$

We can show the following facts.

- ${}^E\mathscr{O}_{L^p}(U) = \{f \in {}^E\mathscr{O}(U \cap \mathbb{C}^n) \mid \forall \rho \in \mathcal{N}(E), \forall K \Subset U, \|f\|_{L^p,\rho,K} < +\infty\}$, where $\|f\|_{L^p,\rho,K}$ is the abbreviation of $\|\rho \circ f\|_{L^p(K \cap \mathbb{C}^n)}$.
- ${}^E\mathscr{O}_{L^p}|_{\mathbb{C}} = {}^E\mathscr{O}$.

- ${}^{E}\mathscr{O}_{L^p}(U) \subset {}^{E}\mathscr{O}_{L^q}(U)$ if $1 \le p \le q \le +\infty$.

We endow ${}^{E}\mathscr{O}_{L^p}(U)$ with a locally convex topology given by a family of semi-norms $\{\|\cdot\|_{L^p,\rho,K}\}_{\rho\in\mathcal{N}(E),K\Subset U}$.

Also introducing another type of semi-norms:

$$\|f\|_{L^\infty L^p,\rho,K} := \sup_{s\in\mathbb{R}^n} \|(\chi_{K\cap\mathbb{C}^n}\rho\circ f)(\cdot+is)\|_{L^p(\mathbb{R}^n)},$$

we have a parallel result with Lemma 3.2.

Lemma 4.2. ${}^{E}\mathscr{O}_{L^p}(U) = \{f \in {}^{E}\mathscr{O}(U\cap\mathbb{C}^n) \mid \forall\rho\in\mathcal{N}(E), \forall K \Subset U, \|f\|_{L^\infty L^p,\rho,K} < +\infty\}$ *as locally convex spaces.*

Corollary 4.3. *Let $S \subset \mathbb{R}^n$ be an open set. For a tube domain $\mathbb{D}^n + iS$, we have*

$$\begin{aligned} {}^{E}\mathscr{O}_{L^p}(\mathbb{D}^n+iS) = \{f \in {}^{E}\mathscr{O}(\mathbb{R}^n+iS) \mid \forall\rho\in\mathcal{N}(E), \forall S_0 \Subset S, \|\rho\circ f\|_{L^\infty(\mathbb{R}^n+iS_0)} < +\infty, \\ \sup_{s\in S_0}\|\rho\circ f(\cdot+is)\|_{L^p(\mathbb{R}^n)} < +\infty\}, \end{aligned}$$

as locally convex spaces.

Then we give vector valued variants of $\mathcal{A}_{L^q}$. (Cf. Lemma 3.4.)

Definition 4.4. ${}^{E}\mathcal{A}_{L^q} := \varinjlim_{d>0} {}^{E}\mathscr{O}_{L^q}(\mathbb{D}^n + i(B_d)^n)$.

In the sequel, we always assume that E is a reflexive locally convex space, and denote by E' its strong dual space. Since the sequential completeness follows from the reflexivity, we can consider the notion of E-valued (and E'-valued) holomorphic functions. Also consider a tube domain $\mathbb{D}^n + iS$ with a connected open set $S \subset \mathbb{R}^n$. The following lemma is a reflexive valued variant of the case $p = +\infty$ and $q = 1$ of Lemma 3.5.

Lemma 4.5. *For $F \in {}^{E'}\mathscr{O}_{L^\infty}(\mathbb{D}^n+iS)$, $f \in {}^{E}\mathscr{O}_{L^1}(\mathbb{D}^n+iS)$, and $s \in S$, the integral $\int_{\mathbb{R}^n} F(t+is)(f(t+is))dt$ is well defined and independent of s.*

Proof. We can easily see that the function $w \mapsto F(w)(f(w))$ is holomorphic.

For an arbitrary $S_0 \Subset S$, the image $\mathcal{M} := F(\mathbb{R}^n + iS_0)$ is a bounded set in E'. Since E is reflexive, E is isomorphic to the strong dual space of E', and

$$\rho_{\mathcal{M}} : E \ni x \mapsto \sup_{y\in\mathcal{M}} |y(x)| \in \mathbb{R}$$

becomes a continuous semi-norm of E. Therefore, it follows from the very definition of ${}^{E}\mathscr{O}_{L^1}(\mathbb{D}^n + iS_0)$, that $\|\rho_{\mathcal{M}}(f(\cdot+is))\|_{L^1(\mathbb{R}^n)}$ and $\|\rho_{\mathcal{M}}(f(\cdot+is))\|_{L^\infty(\mathbb{R}^n)}$ are finite and uniformly bounded in $s \in S_0$.

Note moreover that $|F(w)(f(w))| \leq \rho_{\mathcal{M}}(f(w))$ for any $w \in \mathbb{R}^n + iS_0$. Thus, the function $t \mapsto F(t+is)(f(t+is))$ belongs to $L^1(\mathbb{R}^n)$ for any s, and $w \mapsto |F(w)(f(w))|$ is bounded in $\mathbb{R}^n + iS_0$.

Remaining parts are the same as in Lemma 3.5, the scalar valued case. $\square$

Definition 4.6 (${}^{E'}\mathcal{B}_{L^\infty}$). Let E be a reflexive locally convex space and E' its strong dual space. We define ${}^{E'}\mathcal{B}_{L^\infty} := ({}^{E}\mathcal{A}_{L^1})'$.

We use the same notations U, $\dot{U}$, $\dot{U}_j$ $(j = 1, \dots, n)$, and contours $\gamma(r)$, $\gamma(r,n)$, as in (3.1) and (3.2).

Consider $F \in {}^{E'}\mathscr{O}_{L^\infty}(\dot{U})$ and $\varphi \in {}^{E}\mathscr{O}_{L^1}(U)$. We take r with $0 < r < d$ and define

$$\langle F, \varphi \rangle := \int_{\gamma(r,n)} F(w)(\varphi(w))dw. \tag{4.1}$$

We can prove the well-definedness (Lemma 4.7 below) in a parallel manner as in the scalar valued case.

Lemma 4.7. *The right hand side of* (4.1) *is integrable and independent of* r.

We define the boundary value map $b\colon {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}) \to {}^{E'}\mathcal{B}_{L^\infty}$ by

$$b(F)(\varphi) = \langle F, \varphi \rangle, \quad \text{for } F \in {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}) \text{ and } \varphi \in {}^{E}\mathscr{O}_{L^1}(U),$$

and the defining function map $g\colon {}^{E'}\mathcal{B}_{L^\infty} \to {}^{E'}\mathscr{O}_{L^\infty}(\dot{U})$ by

$$g(\psi)(w)(x) := \psi(K(w - \cdot)x), \quad \text{for } \psi \in ({}^{E}\mathcal{A}_{L^1})',\ w \in \dot{U} \text{ and } x \in E,$$

where K is the function given in (3.5).

Then we have,

- $b(\sum_{j=1}^n {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}_j)) = 0$, and therefore b induces a linear map from the quotient space ${}^{E'}\mathscr{O}_{L^\infty}(\dot{U})/\sum_{j=1}^n {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}_j)$ to ${}^{E'}\mathcal{B}_{L^\infty}$.
- $K(w, \cdot)x$ belongs to ${}^{E}\mathcal{A}_{L^1}$ for a fixed $w \in \dot{U}$ and $x \in E$, and therefore $g(\psi)(w)(x)$ is well-defined.
- $E \ni x \mapsto g(\psi)(w)(x) \in \mathbb{C}$ is linear and continuous.
- $g(\psi)(w)$ is holomorphic in w, and defines a section $g(\psi) \in {}^{E}\mathscr{O}_{L^\infty}(\dot{U})$.

We denote by $[F]$ the class in ${}^{E'}\mathscr{O}_{L^\infty}(\dot{U})/\sum_{j=1}^n {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}_j)$ represented by $F \in {}^{E'}\mathscr{O}_{L^\infty}(\dot{U})$, and we give

Theorem 4.8. *Let E be a reflexive locally convex space. We have $[g(b(F))] = [F]$ for any $F \in {}^{E'}\mathscr{O}_{L^\infty}(\dot{U})$ and $b(g(\psi)) = \psi$ for any $\psi \in {}^{E'}\mathcal{B}_{L^\infty}$. Therefore, b induces an isomorphism between vector spaces:*

$$\frac{{}^{E'}\mathscr{O}_{L^\infty}(\dot{U})}{\sum_{j=1}^n {}^{E'}\mathscr{O}_{L^\infty}(\dot{U}_j)} \xrightarrow{\sim} {}^{E'}\mathcal{B}_{L^\infty}. \tag{4.2}$$

Consider the special case $n = 1$. Then, since $\dot{U}_1 = U$, the left hand side of (4.2) is isomorphic to ${}^{E}\mathscr{B}_{L^\infty}(\mathbb{D}^1)$, as we saw in (2.3). Therefore, (4.2) can be understood as a duality representation for the global sections of E'-valued bounded hyperfunctions at infinity.

Corollary 4.9. *If E is reflexive, then we have, ${}^{E'}\mathscr{B}_{L^\infty}(\mathbb{D}^1) \simeq ({}^{E}\mathcal{A}_{L^1})'$.*

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